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POSITIVELY-CURVED HYPERSURFACES OF A HILBERT SPACE

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1. Statement of the results

A *Riemannian Hilbert manifold* M is a differentiable (C^∞), connected manifold, modeled on a (separable) Hilbert space, such that in the tangent space M_p of M at each $p \in M$ there exists an inner product $\langle , \rangle_p$, which varies differentiably with p (see [7] for precise definitions). M can be made into a metric space by defining the distance between two points $p, q \in M$ as the infimum of the length of differentiable curves joining p and q; M is said to be *complete* if it is complete in this metric.

The local differential geometry of Riemannian Hilbert manifolds develops in exactly the same way as in the finite dimensional case so that we can define a unique covariant derivative and obtain the notions of curvature tensor, geodesics, sectional curvature, etc. (see [8] for details). It can be proved, for example, that convex neighborhoods exist for each point of M [8, pp. 14–16].

However, for the global differential geometry, the situation is quite different. Only a few theorems are known, the main reason being that completeness does not imply, as it does in the finite dimensional case, that two given points of M can be joined by a minimal geodesic; a simple example is given in [3].

The objective of this paper is to prove a global result, for the statement of which we need a few definitions.

A differentiable (C^∞) map $x: M \to H$ of a Riemannian Hilbert manifold M into a Hilbert space H is an *immersion* if the differential $dx(p): M_p \to H$ is one-one and $dx(p)(M_p) \subset H$ is closed in H. If x is one-one it is called an *embedding*.

An *isometric immersion* is an immersion $x: M \to H$ such that $dx(p): M_p \to H$ is an isometry for each $p \in M$. If, in this situation, $dx(p)(M_p) \subset H$ has codimension one, we say that $x(M) \subset H$ is a *hypersurface* of H. Of course, a hypersurface may have self-intersections.

We now state the theorem, which will be proved in §4.

Theorem. *Let M be a complete Riemannian Hilbert manifold with positive sectional curvature K bounded away from zero at each point of M, i.e., for*

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each $p \in M$, *there exists a* $\delta_p > 0$ *such that* $K(\sigma_p) > \delta_p$ *for any two dimensional subspace* $\sigma_p \subset M_p$. *Let* $x: M \to H$ *be an isometric immersion of* M *as a hypersurface of a Hilbert space* H. *Then:*

i) *x is an embedding;*

ii) *$x(M) \subset H$ is the boundary of a convex body in H; in particular, x embeds M topologically as a closed subset of H,*

iii) *M is homeomorphic to a Hilbert space.*

Remark 1. A theorem of this type was first considered by Hadamard [4], who obtained conclusions i) and ii) for the case where $\dim M = 2$, $H = R^3$, and M is compact with positive curvature. The result was later generalized by Stoker [11], allowing M to be complete but keeping the other conditions. The case, where $\dim M = 2$, $H = R^3$, and M is compact with nonnegative curvature, was first treated by Chern and Lashoff in [1]. Finally, Sacksteder [10], using some results of Heijenoort [5], obtained i) and ii) for the case where $\dim M = n$, $H = R^{n+1}$, and M is complete under the conditions that the sectional curvatures of M are nonnegative and, at least at one point, are all positive.

Remark 2. The theorem is likely to be true under weaker conditions on the curvature. In the finite dimensional case, the Sacksteder's theorem quoted at the end of Remark 1 is stronger than our Theorem, and the author does not know of any counter example to a similar statement in the infinite dimensional case.

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2. Notation and preliminary lemmas

M will always denote a Riemannian Hilbert manifold, and H a Hilbert space with inner product $\langle , \rangle$. B_ν denotes the bundle of unit normal vectors of an immersion $x: M \to H$, i.e., a point of B_ν is a pair $(p, \nu(p))$, where $p \in M$ and $\nu(p)$ is a normal vector at $x(p)$. Let $\sum$ be the unit sphere of H, and define a map $\tilde{\nu}: B_\nu \to \sum$ by $\tilde{\nu}(p, \nu(p)) = \nu(p)$. For each $\nu \in \sum$, we also define a *height function* $h: M \to R$ by $h(p) = \langle x(p), \nu \rangle$, $p \in M$.

For any differentiable function $f: M \to R$, $\operatorname{grad} f$ is the vector field in M defined by

$$df(p) \cdot v = \langle \operatorname{grad} f(p), v \rangle_p , \quad p \in M, v \in M_p .$$

A *trajectory* of $\operatorname{grad} f$ is a curve $\varphi(t)$ in M with $\dfrac{d\varphi}{dt} = \operatorname{grad} f(\varphi(t))$, and a *critical point* of f is a point $p \in M$ where $\operatorname{grad} f(p) = 0$. Choosing a coordinate system around a critical point p, the second derivative $d^2f(p)$ defines a symmetric bilinear form on M_p, which does not depend on the coordinate system and is called the *hessian* of f at p. If the self-adjoint linear map of M_p corresponding to the hessian is non-singular, p is a *nondegenerate* critical point. The *index* of a nondegenerate critical point p is the supremum of the dimensions of the subspaces of M_p on which the hessian is negative.

Lemma 1. *Let M be complete, $x: M \to H$ an isometric immersion, and $\varphi(t)$ a trajectory of the gradient field of a height function $h(p) = \langle x(p), \nu\rangle$, $p \in M$. Then $\varphi(t)$ is defined for all $t \in R$.*

Proof. We first show that $\|\operatorname{grad} h\| \leq 1$. By the definition of grad h, we have

$$(1)\qquad \frac{d}{dt}(h \circ \varphi(t)) = dh(\varphi'(t)) = dh(\operatorname{grad} h) = \|\operatorname{grad} h\|^2 .$$

On the other hand, since $h = \langle x, \nu\rangle$ and x is a local isometry,

$$(2)\qquad \frac{d}{dt}(h \circ \varphi(t)) = \langle dx(\varphi'(t)), \nu\rangle = \langle dx(\operatorname{grad} h), \nu\rangle = \langle \operatorname{grad} h, \nu\rangle .$$

Comparing the right hand members of (1) and (2), we obtain the stated inequality.

Now suppose, for instance, that $\varphi(t)$ is defined for $t < t_0$ but not for $t = t_0$. Then, there exists a sequence $\{t_i\}$, $i = 1, \cdots, n, \cdots$ converging to t_0 such that $\{\varphi(t_i)\}$ does not converge. Since $\|\operatorname{grad} h\| \leq 1$, we obtain

$$d(\varphi(t_i), \varphi(t_j)) \leq \int_{t_i}^{t_j} \|\operatorname{grad} h(\varphi(t))\| dt \leq |t_i - t_j| ,$$

where d is the distance in the intrinsic metric of M. It follows that the nonconvergent sequence $\{\varphi(t_i)\}$ is a Cauchy sequence; this contradicts the completeness of M and proves the lemma.

Lemma 2. *Under the same hypotheses of the theorem, except the completeness of M, let $h = \langle x, \nu\rangle$ be a height function on M. Then p is a critical point of h if and only if ν is a normal vector at $x(p)$. In this case, p is a nondegenerate critical point, which is either a maximum or a minimum.*

Proof. $p \in M$ is a critical point of h if and only if, for all $v \in M_p$,

$$dh(p) \cdot v = \langle dx(p) \cdot v, \nu\rangle = 0 ,$$

which proves the first statement.

Now, we choose a coordinate system at p, and write the quadratic form corresponding to the hessian of h at p as

$$(3)\qquad d^2h(p)(v) = \langle d^2x(p)(v), \nu\rangle , \quad v \in M_p .$$

The right hand member of (3) is the normal curvature of $x(M)$ at $x(p)$ in the direction $dx(v)$, and has a fixed sign for all $v \in M_p$ since all sectional curvatures are positive. If we denote by $A: M_p \to M_p$ the self-adjoint map corresponding to the hessian, this means that $\langle Av, v\rangle$, $v \in M_p$, has a fixed sign, and the spectrum of A does not contain zero since the curvature is bounded away from zero at p. It follows that A is non-singular, and hence p is a nondegenerate critical point. Thus it is clear that the index of h at p is either zero or the dimension of M_p; this means that p is either a minimum or a maximum, and hence finishes the proof.

Lemma 3. *Let $x: M \to H$ be an isometric immersion of M as a hypersurface of H. Let $h = \langle x, \nu_0\rangle$ be a height function and $\varphi(t)$, $t \in [a, b]$, be a trajectory of* grad h. *If there exists a point $t_1 \in (a, b)$ such that the function* $\|\text{grad}\, h\varphi(t)\|$ *has a relative minimum for $t = t_1$, then the normal curvature of $x(M)$ at $(\varphi(t_1))$ in the direction of $dx(\varphi'(t_1))$ is zero.*

Proof. Set $\varphi(t_1) = p \in M$, and let W be a neighborhood of p in M such that there exists a differentiable cross-section of the unit normal bundle B_ν over W. Restricting this section to $\varphi(t)$, we obtain a unit normal vector $\nu(t)$ defined in an open interval $J \subset [a, b]$, $t_1 \in J$. For each $t \in J$, let $U_t \subset W$ be a neighborhood of the level surface of h passing through $\varphi(t)$, and let $T_t \subset dx(M_{\varphi(t)}) \subset H$ be the tangent space of $x(U_t)$ at $x(\varphi(t))$.

For notational convenience, we identify vectors in $M_{\varphi(t)}$ with their images in $dx(M_{\varphi(t)}) \subset H$. We remark that T_t has codimension two, and that the vectors grad $h(\varphi(t))$, ν_0, $\nu(t)$ belong to $T_t^\perp$, the orthogonal complement of T_t in H; furthermore $\langle \text{grad}\, h(\varphi(t)), \nu(t)\rangle = 0$.

Next, define $\nu_1(t)$, $t \in J$, as the unit vector of the projection of $\nu(t)$ in $T_t^\perp$ onto the direction normal to ν_0, i.e., $\nu_1(t)$ is differentiable for $t \in J$, and $\langle \nu_1(t), \nu_0\rangle = 0$. Finally, set $\langle \nu(t), \nu_1(t)\rangle = \alpha(t)$, $t \in J$.

Now, we first notice that

$$\|\text{grad}\, h\|^2 = \langle \text{grad}\, h, \nu_0\rangle = \pm\langle \nu(t), \nu_1(t)\rangle = \pm\alpha(t)\,.$$

On the other hand, since $\langle \nu_0, \nu_1(t)\rangle = 0$ and $\langle \nu_1(t), \nu_1(t)\rangle = 1$, we obtain

$$\left\langle \frac{d\nu_1}{dt}, \nu_0\right\rangle = 0\,, \quad \left\langle \frac{d\nu_1}{dt}, \nu_1(t)\right\rangle = 0\,,$$

and hence $\dfrac{d\nu_1}{dt} \in T_t$ and $\left\langle \dfrac{d\nu_1}{dt}, \nu(t)\right\rangle = 0$, It follows that

$$\frac{d\alpha}{dt} = \left\langle \frac{d\nu}{dt}, \nu_1(t)\right\rangle\,.$$

The assumption that $\|\text{grad}\, h(\varphi(t))\| \leq 1$ has a minimum for $t = t_1$ implies that $\dfrac{d\alpha}{dt}(t_1) = 0$ and $|\alpha(t_1)| \neq 1$. Therefore, $\left\langle \dfrac{d\nu}{dt}, \nu_1(t)\right\rangle = 0$ for $t = t_1$, and

$\nu(t_1)$, $\nu_1(t_1)$ are linearly independent. Since $\langle \nu(t), \nu(t) \rangle = 1$, we have also that $\left\langle \frac{d\nu}{dt}, \nu(t) \right\rangle = 0$, from which it follows that $\frac{d\nu}{dt}(t_1) \in T_{t_1}$.

Finally, since $\langle dx(\phi'(t)), \nu(t) \rangle = 0$ and $\frac{d\nu}{dt}(t_1) \in T_{t_1}$, we obtain that $\langle d^2x(\phi'(t)), \nu(t) \rangle = 0$ for $t = t_1$, which proves the lemma.

The following lemma is well known.

Lemma 4. *Let $f\colon M \to R$ be a differentiable function and $p \in M$ a non-degenerate critical point of f, which is either a maximum or a minimum. Then:*

a) *there exists a neighborhood V of p in M such that either $\lim_{t \to -\infty} \phi(t) = p$ (for minimum p) or $\lim_{t \to \infty} \phi(t) = p$ (for maximum p), where $\phi(t)$ is a trajectory of* grad f *with $\phi(0) \in V$;*

b)
$$\inf_{q \in S} \|\text{grad}\, f(q)\| \neq 0\,,$$

if S is a level surface of f sufficiently close to p.

3. The main lemma

Lemma 5. *Under the hypotheses of the Theorem let $h = \langle x, v \rangle$ be a height function on M. Then h has at most two critical points.*

Proof. Let p be a critical point of M. Then by Lemma 2 we may assume, for definiteness, that p is a minimum. Let $\varphi(t)$ be a trajectory of grad h issuing from p, i.e., $\varphi(0)$ is close to p and $\lim_{t \to -\infty} \varphi(t) = p$. Then by Lemma 1, $\varphi(t)$ is defined for all $t > 0$, and we have the following alternatives:

(i) $\lim_{t \to \infty} \varphi(t) = q \in M$. q is then a critical point of h, and we say that $\varphi(t)$ is going into q.

(ii) $\lim_{t \to \infty} \varphi(t)$ does not exist.

Suppose that (i) holds and let S be a level surface of h, sufficiently close to p. Let $A \subset S$ be the set of points in S, which lie on the trajectories of grad h issuing from p and going into q. By continuity, the fact that q is a non-degenerate maximum, and Lemma 4, A is an open set in S. We want to show that A is also closed in S.

Assume that A is not closed in S. Then there exists a sequence of points $\{p_i\}$, $i = 1, \cdots, n, \cdots$, converging to p_0 such that the trajectories φ_i through p_i go into q, but the trajectory φ_0 through p_0 does not. By the above argument, it is clear that φ_0 does not go into another critical point distinct from q. Moreover, by continuity, arbitrarily near to any point of φ_0 there passes a trajectory φ_N for N sufficiently large. It follows that h is bounded on φ_0, and hence $\|\text{grad}\, h\|$ is not bounded away from zero on φ_0.

Now consider small level surfaces S_1 and S_2 of h around p and q respectively.

Since φ_0 does not go into q, we may choose S_2 such that no point of φ_0 belongs to the region B_2 of M bounded by S_2. Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be the infima of $\|\text{grad } h\|$ on S_1 and S_2 respectively, and let $\varepsilon < \min(\varepsilon_1, \varepsilon_2)$, $\varepsilon > 0$ (cf. Lemma 4(b)). By continuity and the fact that $\|\text{grad } h\|$ is not bounded away from zero on φ_0, we can choose a trajectory φ_N for a sufficiently large N such that $\|\text{grad } h(r)\| < \varepsilon$ for some point r on φ_N with $r \notin B_2$ and $r \notin B_1$, where B_1 is the region bounded by S_1. By a translation of the parameter t, we may assume that $\varphi_N(0) \in B_1$ and

$$\frac{d}{dt}\|\text{grad } h \circ \varphi_N(0)\| > 0 .$$

Then the positive differentiable function $\|\text{grad } h \circ \varphi_N(t)\|$, $t \in [0, +\infty)$, changes at least three times from increasing to decreasing or reversely. Therefore this function has a point of relative minimum, which by Lemma 3 implies that the normal curvature of $x(M)$ at such a point in the direction of grad h is zero. This contradicts the hypothesis of the Theorem, and hence A is closed as we claimed.

Since S is connected, $A = S$ and all the trajectories issuing from p go into q.

Now, suppose that (ii) holds. As we have seen, this implies that no trajectory issuing from p has a limit as $t \to \infty$. We want to show that M is then completely covered by the trajectories issuing from p.

The set $A \subset M$ covered by the trajectories issuing from p is, by continuity, an open set in M. To prove that A is closed, we consider a sequence $\{p_i\}$, $i = 1, \cdots, n, \cdots, p_i \in M$, converging to $p_0 \in M$, such that p_i belongs to a trajectory φ_i issuing from p; we will show that the same happens with p_0.

It is clear that p_0 is not a critical point of h, otherwise a trajectory φ_i sufficiently close to p_0 would go into p_0; this contradicts (ii). Therefore, there is a trajectory φ_0 of grad h passing through p_0. We may assume that $\varphi_j(0) = p_j$, $j = 0, 1, \cdots, n, \cdots$, and want to prove that $\lim_{t \to -\infty} \varphi_0(t) = p$.

We remark that arbitrarily near to any point of $\varphi_0((-\infty, 0])$ by continuity there passes a trajectory φ_N, for a sufficiently large N, which goes into p as $t \to -\infty$. It follows that h is bounded on $\varphi_0((-\infty, 0])$, and hence $\|\text{grad } h\|$ is not bounded away from zero on $\varphi_0((-\infty, 0])$.

We now assume that $\varphi_0(t)$ does not go into p as $t \to -\infty$, and choose a level surface S_1 near p such that no point of $\varphi_0((-\infty, 0])$ belongs to the region B_1 bounded by S_1. Let ε_1 be the infimum of $\|\text{grad } h\|$ on S_1, $\varepsilon_2 = \|\text{grad } h(p_0)\|$ and $\varepsilon < \min(\varepsilon_1, \varepsilon_2)$, $\varepsilon > 0$. Arguing as in case (i), we find a trajectory φ_N for a large N such that the function $\|\text{grad } h \circ \varphi_N(t)\|$, $t \in (-\infty, 0]$, has a point of minimum. Application of Lemma 3 shows that our assumption leads to a contradiction; hence A is closed in M.

By connectedness, $A = M$ and M is completely covered by the trajectories issuing from p.

An entirely similar argument shows that also in case (i), M is completely covered by the trajectories issuing from p and going into q. Therefore h has at most two critical points, and hence the proof of Lemma 5 is finished.

Remark. Actually, in view of the first part of Lemma 2, we have proved a little more, namely: Either h has one minimum (or maximum) p and the trajectories of grad h issuing from p cover M, or h has one minimum p and one maximum q and the trajectories of grad h issuing from p and going into q cover M.

4. Proof of the Theorem

Proof of (i). Suppose that p, q are distinct points in M with $x(p) = x(q)$. Consider the height function $h = \langle x, \nu\rangle$, where ν is a normal vector p. By Lemma 2, p is a critical point of h and, by Lemma 5 there is a trajectory of grad h issuing from p and passing through (or going into) q. Thus $x(p)$ and $x(q)$ are at different levels; this contradicts the fact that $x(p) = x(q)$ and proves (i).

Proof of (ii). The above argument shows that $x(M)$ lies entirely on one side of the tangent hyperplane of $x(M)$ at $x(p)$ for any $p \in M$, and has no point in common with this hyperplane other than $x(p)$.

Now, the intersection of all closed half spaces of H, which are bounded by the tangent hyperplanes and contain points of $x(M)$, is a closed convex subset $K \subset H$. It is clear that $x(M)$ is contained in the boundary K' of K. Moreover, since the curvature is positive, $x(M)$ is not contained in a hyperplane, and K has interior points. Therefore K is a convex body in H.

In order to prove that $x(M) = K'$, we first show that $x(M)$ is an open set in K'. For this purpose, we remark that through each point k of K' there passes a support hyperplane of K, i.e., a hyperplane of H containing points of the closure of K but no interior points of K. This follows from a geometric form of the Hahn-Banach theorem, which implies that k and K are separated by a hyperplane (see [2, p. 417]).

Now, since $x: M \to H$ is an immersion, for each $p \in M$ there exists a neighborhood U of p in M with $x(U) \subset K'$. We will show that there is a neighborhood V of $x(p)$ in H such that $V \cap K' = x(U)$. Assuming the contrary, we obtain a sequence $\{k_i\}$, $k_i \in K'$, $k_i \notin x(U)$, $i = 1, \cdots, n, \cdots$, which converges to $x(p)$. Let π be the tangent hyperplane of $x(M)$ at $x(p)$. Clearly the sets $x(U)$ and $\{k_i\}$ are on the same side of π, and $k_i \notin \pi$. By the above remark, through each k_i there passes a support hyperplane π_i of K. If all π_i are parallel, then there will be points of $x(U)$ on both sides of some π_i; if some π_i for large i is not parallel to π, then it will intersect $x(U)$. In any case, our assumptions lead to a contradiction, which shows that $x(M)$ is an open set in K'.

We now prove that $x(M) = K'$. Assume that there exists a point $k \in K'$, $k \notin x(M)$, and fix any point $x(p) \in x(M)$. Since K' is the boundary of a convex body, K' is connected [6, p. 31] and there exists a rectifiable curve $\theta(t)$ in

K', $t \in [0, a]$, with $\theta(0) = x(p)$ and $\theta(a) = k$. Since $x(M)$ is open in K', there is a neighborhood of $x(p)$ containing an initial segment of θ, and therefore exists a point $\xi \in [0, a]$ such that $\theta(t) \in x(M)$ for $t \in [0, \xi)$ and $\theta(\xi) \notin x(M)$. This implies the existence of a sequence $\{t_i\}$, $t_i < \xi$, $i = 1, \cdots, n, \cdots$, converging to ξ such that the sequence $\{\theta(t_i)\}$ does not converge in $x(M)$. We denote by a_i the length of $\theta(t)$ from 0 to t_i, and remark that the sequence $\{a_i\}$ converges. From this and the inequality

$$d(\theta(t_i), \theta(t_j)) \leq |a_i - a_j| ,$$

due to the local isometry of x, we conclude that $\{\theta(t_i)\}$ is a nonconvergent Cauchy sequence; this contradicts the completeness of M and proves (ii).

Proof of (iii). We first show that the convex body K does not contain an entire line of H. Assume that there exists a line $L \subset K$ and let π be a hyperplane tangent to $x(M)$ at $x(p)$. Then by convexity, the segment $\overline{x(p)r}$ joining $x(p)$ to any point $r \in L$ belongs to K. The limit points of all segments $\overline{x(p)r}$, $r \in L$, fill up a line L' parallel to L and passing through p, and hence are contained in π. Since K is closed, $L' \subset K$. It follows that the tangent hyperplane π contains an entire line of points of K; this contradicts the remark made at the beginning of the proof of (ii).

By the above argument and the topological classification of boundaries of convex bodies in a Hilbert space [6, p. 31, Prop. 1.7], we obtain (iii), and the proof of the Theorem is complete.

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